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MULTICOMMODITY STEADY STATE INVENTORY MODEL SUBJECT TO LINEAR RESTRICTIONS

by

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ABSTRACT

This paper is concerned with (i) the maximization of net return $\sum_{j=1}^{n} \pi_j(y_j, t_j)$ and, (ii) the maximization of average net return $\sum_{j=1}^{n} \pi_j(y_j, t)/t$ of a deterministic multicommodity system subject to linear restrictions on the inventory levels y_j and the review periods t_j $(j=1, \ldots, n)$.
The first problem is a symmetric quadratic program. For the second problem, an algorithm is given for finding the optimal solution.

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I. INTRODUCTION

1. In this paper is considered: the maximization of the net return $\sum \pi_j(y_j,t_j)$ and the average net return $\sum \pi_j(y_j,t)_{/t}$ of a deterministic multicommodity system subject to linear restrictions on the inventory levels y_j and the review periods t_j.

2. It will be useful to give first, a short description of the deterministic multicommodity system whose the net return is $\pi(y,t)$. The characteristics of such a system are the following [7].

(i) Demand is deterministic at a constant rate of R quantity units per time unit, $R > 0$.

(ii) The replenishment rate is infinite.

(iii) h = inventory holding cost, $h > 0$, dimension: $\frac{\$}{\text{Unit.Unit-time}}$.

(iv) c = cost of production, dimension: $\frac{\$}{\text{Unit}}$

(v) r = revenue, dimension: \$

(vi) K = ordering cost, dimension: \$

(vii) $\bar{P}$ = penalty (shortage cost), dimension: $\frac{\$}{\text{Unit.Unit-Short}}$

(viii) P = penalty (good will lost), dimension: $\frac{\$}{\text{Unit-Short}}$

(ix) y = inventory level, dimension: Unit.

(x) t = review period time, dimension: Time

(xi) $\pi(y,t)$ = net revenue, dimension: \$

(xii) h, c, r, k, $\bar{P}$ and P are nonnegative constans, whereas y and t are variables.

The value of the net return $\pi(y,t)$ is given by

$$\pi(y,t) = \begin{cases} F(y,t) & \text{if} \quad y \leqq 0 \\ G(y,t) & \text{if} \quad 0 \leqq y \leqq Rt \\ H(y,t) & \text{if} \quad y \geqq Rt \end{cases} \tag{1}$$

where:

$$(2) \qquad F(y,t) = (r-c)Rt - \left[K + \bar{p}\left(\frac{Rt}{2} - y\right) + pRt\right]$$

$$(3) \qquad G(y,t) = (r-c)Rt - \left[K + \frac{h}{2R}\,y^2 + \frac{\bar{P}}{2R}\,(Rt-y)^2 + P(Rt-y)\right]$$

$$(4) \qquad H(y,t) = (r-c)Rt - \left[K + h\left(y - \frac{R}{2}\,t\right)t\right]$$

From (2), (3) and (4) it can be observed that $\pi(y,t)$ has the following properties:

(a) $\pi(y,t)$ is a continuous function of y (unrestricted) and $t \geq 0$.

(b) $\frac{\partial\pi(y,t)}{\partial y}$ exists and is a continuous function except at the points $(0,t)$ and (Rt,t) .

(c) $\frac{\partial\pi(y,t)}{\partial t}$ exists and is a continuous function except at the points (Rt,t) .

(d) $\pi(y,t)$ is a concave function for all y and t such that $0 \leq y \leq Rt$.

Define the negative net return $N(y,t)$

$$(5) \qquad N(y,t) = -\pi(y,t)$$

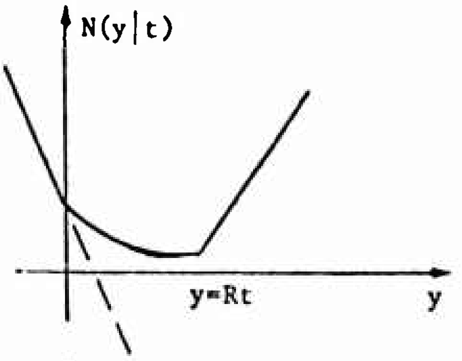
$$(6) \qquad N(y|t) = N(y,t = \text{constant})$$

From (2), (3) and (4)

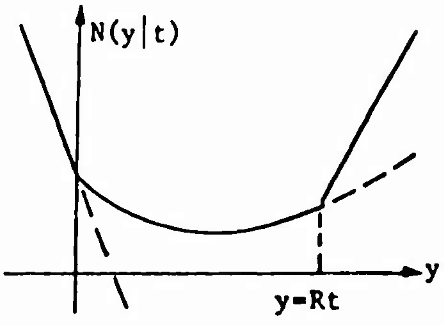
$$(7) \qquad \frac{\partial N(y|t)}{\partial y} = \begin{cases} \bar{p}t & \text{if} \quad y < 0 \\ \frac{h}{R}\,y - \frac{\bar{P}}{R}\,(Rt-y) - p & \text{if} \quad 0 \leq y \leq Rt \\ ht & \text{if} \quad y > Rt \end{cases}$$

$$\left.\frac{\partial N(y|t)}{\partial y}\right|_{y=Rt-} = ht-p$$

The graphical representation of $N(y|t)$ is given in Figure 1.



$\left(t < \frac{P}{h}\right)$ Figure 1-a.



$\left(t \geq \frac{P}{h}\right)$ Figure 1-b.

It is clear that the function $N(y|t)$ attains its minimum at a point y^* such that $y^* = Rt$ if $t < \frac{P}{h}$ or $0 \leq y^* < Rt$ if $t \geq \frac{P}{h}$.

Lost Sale Case

In the model discussed before it was assumed that all demands incurred when the system was out of stock were backordered. If demands occurring when the system is out of stock are lost forever, the value of the net return $\pi(y,t)$ will depend on the length of time for which the system is out of stock. In this case, since backorders are not allowed, the penalty due to shortage is null, whereas the penalty due to the good will lost is equal to $r-c$, i.e., $\bar{p} = 0$ and $p = r-c$ [9]; with these changes in the equation (1), the net revenue for the Lost Sale Case becomes

$$\text{(8)} \qquad \pi(y,t) = \begin{cases} 0 & \text{if } y \leq 0 \\ (r-c)y - \left[K + \frac{h}{2R} y^2\right] & \text{if } 0 \leq y \leq Rt \\ (r-c)Rt - \left[K + h\left(y - \frac{R}{2} t\right)t\right] & \text{if } y \geq Rt \end{cases}$$

II. RESTRICTIONS

1. Restrictions on the Inventory Levels

There can be many sorts of interactions between items when an inventory system stocks many items. For example, warehouse capacity may be limited, in which case the items compete for the floorspace; or there maybe an upper limit to the maximum investment in inventory whence the items compete for investing dollars; or the total weight of items that the warehouse floor can tolerate may be limited; etc.

Consider the case where there is an upper limit b_1 to the square feet of warehouse floor space. Suppose that n items are being stocked and one unit of item j takes up b_{1j} square feet space, then if the floor space constraint is not to be violated at any time, it must be true that

$$(9) \qquad b_{11}\,\delta(y_1,0) + \dots + b_{1n}\,\delta(y_n, 0) \leq b_1, \text{ where}$$

$$\delta(y_j,0) = \begin{cases} y_j & \text{if } y_j \geq 0 . \\ 0 & \text{if } y_j < 0 . \end{cases}$$

It must be noted that in restrictions of this sort all coefficients b_{1j} $(j=1, \dots, n)$ are nonnegative and $b_1 > 0$.

2. Restrictions on the Storge Period of the Commodities

Some commodities (e.g. perishable goods) cannot be stored any longer than a certain period of time T . Therefore in such a case must be, for those commodities $t_j \leq T$, where T is a positive constant.

If all replenishments are made up at the same time, then the restrictions are of the form $t_i = t_j$ for all $i,j=1, \dots, n$.

3. Restrictions Involving y_j and $t_j (j=1, \dots, n)$.

In some models all demands must be met from inventory so that no back-orders or lost sales are allowed, in this case the restrictions are $y_j = R_j t_j$ $j=1, \ldots, n$; [7].

If the maximization of $\sum_{j=1}^{n} \pi_j(y_j, t_j)$ is only for one period, then all shortages are lost sales and the optimal solution must be such that $y_j^* \leqq R_j t_j^*$ $j=1, \ldots, n$; that is, $y_j^* > R_j t_j^*$ must not permitted since there is not a second period for selling the difference $y_j^* - R_j t_j^*$. Therefore, $y_j \leqq R_j t_j$ must be a restriction for all index $j=1, \ldots, n$.

III. MAXIMIZATION OF $\sum_{j=1}^{n} \pi_j(y_j, t_j)$.

It is clear that under any set of restrictions, the maximization of $\sum \pi_j(y_j, t_j)$ is equivalent to the minimization of $\sum N_j(y_j, t_j)$. Since the ordering cost $K_j (j=1, \ldots, n)$ is a constant, without lost of generality it can be assumed that $K_j = 0$ $(j=1, \ldots, n)$.

1. The Program Under Consideration Is Program A:

$$
(10) \qquad
\begin{array}{ll}
\text{Minimize} & N(y,t) = \sum_{j=1}^{n} N_j(y_j, t_j) \\
\text{Subject to} & b_{11}\,\delta(y_1,0) + \ldots + b_{1n}\,\delta(y_n,0) \leqq b_1 \\
& \vdots \\
& b_{r1}\,\delta(y_1,0) + \ldots + b_{rn}\,\delta(y_n,0) \leqq b_r \\
& d_{11}t_1 + \ldots + d_{1n}t_n \leqq d_1 \\
& \vdots \\
& d_{s1}t_1 + \ldots + d_{sn}t_n \leqq d_s \\
& y_j \text{ unrestricted, } t_j \geqq 0 \ (j=1, \ldots, n) .
\end{array}
$$

It will be shown that the program A is equivalent to the following

Program A'

$$
\text{(11)} \quad
\begin{aligned}
&\text{Minimize} \quad N(y,t) = \sum_{j=1}^{n} N_j(y_j,t_j) \\
&\text{Subject to} \quad b_{11}y_1 + \ldots + b_{1n}y_n \leqq b_1 \\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\qquad\qquad b_{r1}y_1 + \ldots + b_{rn}y_n \leqq b_r \\
&\qquad\qquad d_{11}t_1 + \ldots + d_{1n}t_n \leqq d_1 \\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\qquad\qquad 0 \leqq y_j \leqq R_j t_j, \ j=1, \ldots, n .
\end{aligned}
$$

where

$y = y(y_1, \ldots, y_n)'$, $t = (t_1, \ldots, t_n)'$, $(y,t) = (y_1, \ldots, y_n; t_1, \ldots, t_n)$

all coefficients of y_j and t_j, and b_i and d_k are constants for all $j=1, \ldots, n$; $i=1, \ldots, r$; $k=1, \ldots, s$. $\delta(y_j,0)$ is defined by (9) and $b_{ij} \geqq 0$, $b_i > 0$ for all $i=1, \ldots, r$; $j=1, \ldots, n$.

Theorem 1

The solution (y^*,t^*) is optimal for the program A if and only if it is optimal for the program A'.

Proof:

Define

$$
\begin{aligned}
N &= \{1, \ldots, n\} . \\
S &= \{(y,t) \mid (y,t) \text{ satisfies (10)}\} . \\
S' &= \{(y,t) \mid (y,t) \text{ satisfies (11)}\} .
\end{aligned}
$$

Let (y^o,t^o) be a solution of (10), i.e., $(y^o,t^o) \varepsilon S$. Associated with (y^o,t^o) defined the sets:

$$N_1 = \{j \mid j\epsilon N,\ y_j^o < 0\} .$$

$$N_2 = \{j \mid j\epsilon N,\ 0 \leqq y_j^o \leqq R_j t_j^o\} .$$

$$N_3 = \{j \mid j\epsilon N,\ y_j^o > R_j t_j^o\} .$$

Let $(y',t^o) = (y_1', \ldots, y_n';\ t_1^o, \ldots, t_n^o)'$ such that

$$y_j' = \begin{cases} 0 & \text{if } j\epsilon N_1 \\ y_j & \text{if } j\epsilon N_2 \\ R_j t_j^o & \text{if } j\epsilon N_3 \end{cases}$$

The vector (y',t^o) satisfies the restriction (11), because for all $i=1, \ldots, r$; it is true that

$$\begin{aligned} b_{ij}y_j' &= b_{ij}\ \delta(y_j^o,0) = 0 && \text{if } j\epsilon N_1 \\ b_{ij}y_j' &= b_{ij}y_j^o && \text{if } j\epsilon N_2 \\ b_{ij}y_j' &\leqq b_{ij}y_j^o && \text{if } j\epsilon N_3 \end{aligned}$$

Furthermore, since $y_j' > y_j^o$ if $j\epsilon N_1$, $y_j' = y_j^o$ if $j\epsilon N_2$ and $y_j' < y_j^o$ if $j\epsilon N_3$, a direct application of (7) gives:

$$\begin{aligned} N_j(y_j',t_j^o) &\leqq N_j(y_j^o,t_j^o) && \text{if } j\epsilon N_1 \\ N_j(y_j',t_j^o) &= N_j(y_j^o,t_j^o) && \text{if } j\epsilon N_2 \\ N_j(y_j',t_j^o) &< N_j(y_j^o,t_j^o) && \text{if } j\epsilon N_3 \end{aligned}$$

That is, for each solution $(y^o,t^o)\epsilon\ S$ there is a solution $(y',t^o)\epsilon\ S'$ such that $N(y',t^o) \leqq N(y^o,t^o)$. This implies:

$$\text{(12)} \qquad \underset{(y,t)\epsilon\ S'}{\text{minimum}}\ N(y,t) \leqq \underset{(y,t)\epsilon\ S}{\text{minimum}}\ N(y,t)$$

On the other hand

$S' \subset S$, implies:

$$\text{(13)} \qquad \underset{(y,t)\in S'}{\text{minimum}} \; N(y,t) \geq \text{minimum} \; N(y,t)$$

From (12) and (13), the Theorem follows.

Q.E.D.

In view of Theorem 1., it will be sufficient to consider the function $N(y,t)$ only on the region $0 \leq y_j \leq R_j t_j$, $j=1, \ldots, n$.

Then, from (3) when $0 \leq y_j \leq R_j t_j$, $K_j = 0$

$$N(y,t) = \sum_{j=1}^{n} N_j(y_j,t_j)$$

$$\text{(14)} \qquad = \sum_{j=1}^{n}\left[\frac{h_j}{2R_j}\, y_j^{\,2} + \frac{\bar{P}_j}{2R_j}\,(R_j t_j - y_j)^2\right] + \sum_{j=1}^{n}[(p_j + c_j - r_j)R_j t_j - P_j y_j]$$

Using matrix notation, the first term on the right hand side can be written in the form

$$\text{(15)} \qquad \sum_{j=1}^{n}\left[\frac{h_j}{2R_j}\, y_j^{\,2} + \frac{\bar{P}_j}{2R_j}\,(R_j t_j - y_j)^2\right] \equiv (y,t)'Q(y,t)$$

where

$(y,t) = (y_1, \ldots, y_n; t_1, \ldots, t_n)'$ and Q is the $2n \times 2n$ matrix

$$Q = \begin{bmatrix} \frac{h_1+\bar{P}_1}{2R_1} & & & -\frac{\bar{P}_1}{2} & & \\ & \ddots & & & \ddots & \\ & & \frac{h_n+\bar{P}_n}{2R_n} & & & -\frac{\bar{P}_n}{2} \\ -\frac{P_1}{2} & & & \frac{R_1\bar{P}_1}{2} & & \\ & \ddots & & & \ddots & \\ & & -\frac{\bar{P}_n}{2} & & & \frac{R_n\bar{P}_n}{2} \end{bmatrix}$$

Observe that the matrix Q is positive (semi-) definite since all coefficients of y_j^2 and $(R_jt_j-y_j)^2$ on the left hand side of (15) are nonnegative. That is, $(y,t)'Q(y,t) \geqq 0$ for each vector (y,t). In particular, the left hand side of (15) is always positive for each vector $(y,t) \neq 0$, if $\bar{P}_j > 0$ for all $j=1, \ldots, n$; i.e., the matrix Q is positive definite if $\bar{P}_j > 0$ for all $j=1, \ldots, n$.

(Note: at the beginning it was assumed $h_j > 0$, $R_j > 0$ and $\bar{P}_j \geqq 0$ for all $j=1, \ldots, n$).

Then, an equivalent formulation of the program A' is

$$\text{(16)} \qquad \begin{array}{ll} \text{Minimize} & N(x) = x'Ax + g'x \\ \text{Subject to} & Ax + b \geqq 0 \\ & x \geqq 0 \end{array}$$

where

$$x = (y,t) = (y_1, \ldots, y_n; t_1, \ldots, t_n)$$

$$g = (-\bar{P}_1, \ldots, -\bar{P}_n; (P_1+c_1-r_1)R_1, \ldots, (P_n+c_n-r_n)R_n)'$$

$$A = \begin{bmatrix} -b_{11} & \cdots & -b_{1n} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -b_{r1} & \cdots & -b_{rn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -b_{11} & \cdots & -d_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -d_{s1} & \cdots & -d_{sn} \\ -1 & & & R_1 & & \\ & \ddots & & & \ddots & \\ & & -1 & & & R_n \end{bmatrix}$$

and b is the $(r+S+n)$ column vector

$$b = (b_1, \ldots, b_r; d_1, \ldots, d_s; 0, \ldots, 0)'$$

2. Existence of the Solutions.

The set $S' = \{(y,t) \mid (y,t)$ satisfies $(11)\}$ is a closed convex set. If in addition it is bounded, then the program A' always has an optimal solution, because $N(y,t)$ is a continuous function and a continuous function defined on a compact (closed and bounded) set has an absolute minimum on it.

Let S_1 and S_2 be such that

$$S_1 = \{y \mid (y,t) \varepsilon\ S'\} \quad \text{and} \quad S_2 = \{t \mid (y,t) \varepsilon\ S'\}$$

The set S_1 is clearly bounded because in the set of restrictions (11) it was assumed that $b_{ij} \geqq 0$ and $b_j > 0$ for all $i=1, \ldots, r; j=1, \ldots, n$. The set S_2 may be unbounded.

However, if $\bar{P}_j > 0$ for all $j=1, \ldots, n$; then the program A' has always an optimal solution. This affirmation follows immediately from the following:

(i) S is a closed convex set, and (ii) when $P_j > 0$ for all $j=1, \ldots, n$; the objective function $N(y,t)$ cannot be unbounded in the direction of extremization, since it is the sum of a positive definite quadratic form and a linear expression

3. There are many ways to obtain the optimal solution of program A', a quadratic symmetric program. One method is the following: See [1].

Find $Z \geqq 0$, such that

$$MZ + q \geqq 0 \quad \text{and}$$
$$Z'(Mz + q) = 0$$

where

$$M = \begin{pmatrix} 2Q & -A' \\ A & 0 \end{pmatrix}; \; q = (g,b) \quad \text{and}$$

$Z = (x,\lambda)$. The components of the vector λ are the nonnegative multipliers associated with the inequalities of (16).

Example 1

A retail merchant in city A makes a weekly trip to city B in order to refill his supplies. The truck he uses, restricts the volume of the goods. Therefore the merchant must decide how much of each commodity he should take such that, (i) the truck restrictions are not violated, and, (ii) his total profit is maximized.

All shortages are lost sales and the set up cost is the cost of the trip.

c_j is the buying price at city B; r_j, h_j and R_j are respectively the selling price, inventory holding cost and demand of items $j=1, \ldots, n$; v_j and w_j are the volume and weight of item j . Finally V and W are the volume and weight capacities of the truck.

For instance, with n=2 and the following values:

Item	r_j	c_j	h_j	R_j	$\bar{P}_j$	P_j	v_j	w_j	K	V	W
1	5	3	0.4	20	0	2	6	3	100	2400	1500
2	3	2	0.5	50	0	1	4	5			

For reason of space all dimensions were dropped.

The unit time is 1 day,

" " volume is 1 cubic foot,

" " weight is 1 pound.

Then, the problem is

$$\text{Minimize} \quad \left[\frac{1}{100} y_1^{\,2} + \frac{1}{200} y_2^{\,2} - 2y_1 - y_2 + 100\right]$$

$$\begin{aligned} \text{Subject to} \quad 3y_1 + 5y_2 &\leq 1500 \\ 6y_1 + 4y_2 &\leq 2400 \\ y_1 &\leq 20\text{x}7 \\ y_2 &\leq 50\text{x}7 \end{aligned}$$

The optimal solution is $y_1 = 100$, $y_2 = 100$.

IV. MAXIMIZATION OF $\sum_{j=1}^{n} \pi_j(y_j,t)/t$

The maximization of the average total net revenue per unit time when all replenishments are made up simultaneously and the inventory levels are subject to linear restrictions is considered in this section.

For convenience, one will minimize $\sum_{j=1}^{n} N_j(y_j,t)/t$ instead of maximizing $\sum_{j=1}^{n} \pi_j(y_j,t)/t$.

1. The Program Under Consideration Is Program B:

$$
\begin{aligned}
\text{Minimize} \quad & N(y,t)/t = \sum_{j=1}^{n} N_j(y_j,t)/t \\
\text{Subject to} \quad & b_{11}\, \delta(y_1,0) + \dots + b_{1n}\, \delta(y_n,0) \leqq b_1 \\
& \vdots \\
& b_{r1}\, \delta(y_1,0) + \dots + b_{rn}\, \delta(y_n,0) \leqq b_r \\
& y_j \text{ is unrestricted } \quad j=1, \dots, n \\
& t \geqq 0 .
\end{aligned}
$$

where

$(y,t) = (y_1, \dots, y_n,t)'$ is an $(N+1)$ column vector and the restriction on the inventory levels y_j are the same as in program A'.

Consider also

Program B':

$$\text{Minimize} \quad N(y,t)_{/t} = \sum_{j=1}^{n} N_j(y_j,t)_{/t}$$

$$\begin{aligned}
\text{Subject to} \quad & b_{11}y_1 + \dots + b_{1n}y_n \leqq b_1 \\
& \vdots \\
& b_{r1}y_1 + \dots + b_{rn}y_n \leqq b_r \\
& 0 \leqq y_j \leqq R_j t;\ j=1, \dots, n \\
& t \geqq 0 .
\end{aligned}$$

Theorem 2

The solution (y^*,t^*) is optimal for the program B if and only if it is optimal of the program B'.

With $t_j=t$ $(j=1, \dots, n)$ the proof of this theorem follows the same line as the one for Theorem 1.

Define

(19) $$T = \{(y,t) \mid (y,t) \text{ satisfies } (18)\} .$$

Theorem 3

If $\frac{N(y^*,t^*)}{t^*} \leqq 0$ and $N(y^+,t^+) < 0$ then $t^* \leqq t^+$.

Where (y^*,t^*) is the optimal solution of program B' and $N(y^+,t^+) =$

$= \underset{(y,t)\varepsilon T}{\text{minimum}} N(y,t)$.

Proof:

If (y^*,t^*) is an optimal solution of program B', then

$$\frac{N(y^*,t^*)}{t^*} \leqq \frac{N(y,t)}{t} \quad \text{for all point } (y,t)\varepsilon T,$$

in particular:

(20) $$\frac{N(y^*,t^*)}{t^*} \leqq \frac{N(y^+,t^+)}{t^+}, \text{ since } (y^+,t^+)\varepsilon\ T .$$

On the other hand,

$$N(y^+,t^+) \leqq N(y,t) \quad \text{for all point} \quad (y,t)\varepsilon\ T,$$

in particular:

(21) $$N(y^+,t^+) \leqq N(y^*,t^*), \text{ since } (y^*,t^*)\varepsilon\ T .$$

Using the hypothesis of the theorem in (20) and (21) one can get, respectively

(22) $$\frac{N(y^*,t^*)}{N(y^+,t^+)} \geqq \frac{t^*}{t^+} \quad \text{and (23)} \quad \frac{N(y^*,t^*)}{N(y^+,t^+)} \leqq 1$$

From (22) and (23)

$$\frac{t^*}{t^+} \leqq \frac{N(y^*,t^*)}{N(y^+,t^+)} \leqq 1 . \quad \text{Hence} \quad t^* \leqq t^+ .$$

Q.E.D.

The economical meaning of this result is: if the optimal total net revenue for the one period analysis is positive and its optimal review period is t^+, when the optimal average total net revenue per unit time is nonnegative, then t^+ is an upper bound for the optimal review period of the Steady State Case.

In view of Theorem 2., it will be sufficient to consider the function $N(y,t)_{/t}$ only on the region $0 \leqq y_j \leqq R_j t$ $(j=1, \ldots, n)$.

From (3).

$$N(y,t)_{/t} = \sum_{j=1}^{n}\left[K_j + \frac{h_j}{2R_j}\, y_j^{\,2} + \frac{\bar{P}_j}{2R_j}\,(R_j t-y_j)^2+(P_j+c_j-r_j)R_j t-P_j y_j\right]/t$$

$$= N'(y,t) + N''(t)$$

where

$$(24) \qquad N'(y,t) = \sum_{j=1}^{n} \left[\frac{h_j+\bar{P}_j}{2R_j t} y_j^{\,2} - \left(\bar{P}_j + \frac{P_j}{t}\right) y_j \right]$$

$$(25) \qquad N''(t) = \sum_{j=1}^{n} \left[\frac{K_j}{t} + \frac{R_j \bar{P}_j}{2} t + (P_j + r_j - c_j) R_j \right]$$

Define

$$(26) \qquad T(t) = \{y \,|\, (y,t) \varepsilon\ T\} \ .$$

Note that $T(t)$ is the set of solutions y of (18) for a fixed value of t, and $T(t)$ is a compact set.

Then, an equivalent formulation of the program B' is

$$\text{Minimize} \quad \frac{N(y,t)}{t}$$

$$\text{Subject to} \quad (y,t) \varepsilon\ T \ .$$

Observe that

$$\begin{aligned}
&\underset{(y,t)\varepsilon T}{\text{Minimum}} \ \frac{N(y,t)}{t} = \\
&\qquad = \underset{t \geqq 0}{\text{minimum}} \left(\underset{y\ \varepsilon\ T(t)}{\text{minimum}} \ \frac{N(y,t)}{t} \right) \\
(27) \qquad &\qquad = \underset{t \geqq 0}{\text{minimum}} \left(\underset{y\ \varepsilon\ T(t)}{\text{minimum}} \ (N'(y,t) + N''(t)) \right) \\
&\qquad = \underset{t \geqq 0}{\text{minimum}} \left(N''(t) + \underset{y\ \varepsilon\ T(t)}{\text{minimum}} \ N'(y,t) \right)
\end{aligned}$$

Define

$$(28) \qquad H(t) = \underset{y\ \varepsilon\ T(t)}{\text{minimum}} \ \frac{N(y,t)}{t} = h(t) + N''(t)$$

$$(29) \qquad h(t) = \underset{y\ \varepsilon\ T(t)}{\text{minimum}} \ N'(y,t)$$

Theorem 4

The program B' always has an optimal solution.

In order to prove Theorem 4., the following definitions and theorems are necessary.

Definition 1. Scalar function

A correspondence which assigns a single scalar (a real number, a point in R') to each point x of a set Γ is called a scalar function of x and is written as $\Theta(x)$. The scalar function $\Theta(x)$ is said to be defined on Γ .

Definition 2. Quasi-convex function

A scalar function $\Theta(x)$ defined on a convex set $\Gamma \subset R^n$ is said to be quasi-convex on Γ if $\Theta(\alpha x + (1-\alpha)y) \leqq \max(\Theta(x), \Theta(y))$ for each pair of points $x, y \in \Gamma$ and all real $\alpha \geqq 0$.

Definition 3. Strictly quasi-convex function

A scalar function $\Theta(x)$ defined on a convex set $\Gamma \subset R^n$ is said to be strictly quasi-convex on Γ if for each pair of points $x, y \in \Gamma$ such that $\theta(x) \neq \Theta(y)$, it is true that $\Theta(\alpha x + (1-\alpha)) < \max(\Theta(x), \Theta(y))$ for all real $\alpha > 0$.

Theorem 5

The program: minimize $N'(y,t)$ such that $y \in T(t)$, always has an optimal solution.

Proof:

Since $N'(y,t)$ is a continuous function and $T(t)$ is a compact set, $N'(y,t)$ has an absolute minimum on $T(t)$.

Theorem 6. [3]

The function

$$M(t) = \underset{y \in T(t)}{\text{minimum}} N(y,t), \text{ is convex}$$

Proof:

The function $N(y,t)$ is continuous and the set $T(t)$ is compact. Therefore for each number $t^o \geq 0$ there is a point $y^o \varepsilon T(t^o)$ such that $N(y^o, t^o)$ is an absolute minimum.

Let $t_1 \geq 0$ and $t_2 \geq 0$, with $t_1 \neq t_2$. Then there exists $y^1 \varepsilon T(t_1)$ and $y^2 \varepsilon T(t_2)$ such that

(30) $$M(t_1) = \underset{y \in T(t_1)}{\text{minimum}} N(y,t_1) = N(y^1,t_1)$$

(31) $$M(t_2) = \underset{y \in t(t_2)}{\text{minimum}} N(y,t_2) = N(y^2,t_2)$$

On the other hand the set of restrictions (19) are linear. Therefore, if $y^1 \varepsilon T(t_1)$ and $y^2 \varepsilon T(t_2)$ then $(\alpha y^1 + (1-\alpha)y^2) \varepsilon T(\alpha t_1 + (1-\alpha)t_2)$ for all real $\alpha \geq 0$.

Hence

$$\begin{aligned} M(\alpha t_1 + (1-\alpha)t_2) &= \underset{y \in T(\alpha t_1 + (1-\alpha)t_2)}{N(y, \alpha t_1 + (1-\alpha)t_2)} \\ &\leq N(\alpha y^1 \quad 1-\alpha)y^2, \alpha t_1 + (1-\alpha)t_2) \end{aligned}$$

But the function $N(y,t)$ is convex, then

$$M(\alpha t_1 + (1-\alpha)t_2) \leq \alpha N(y^1,t_1) + (1-\alpha)N(y^2,t_2) .$$

Using (30) and (31)

$$M(\alpha t_1 + (1-\alpha)t_2) \leq M(t_1) + (1-\alpha)M(t_2) .$$

That is, $M(t)$ is a convex function.

Q.E.D.

Corollary 1. [3]

The function $H(t) = \underset{y \in T(t)}{\text{minimum}} \frac{N(y,t)}{t}$ is quasi-convex and defined on $(0, +\infty)$.

Proof:

$$H(t) = \underset{y \in T(t)}{\text{minimum}} \frac{N(y,t)}{t} = \left(\underset{y \in T(t)}{\text{minimum}} N(y,t)\right)\frac{1}{t} = \frac{M(t)}{t}$$

Then, for all $\alpha \geq 0,\ t_1 > 0,\ t_2 > 0$,

$$H(\alpha t_1 + (1-\alpha)t_2) = \frac{M(\alpha t_1 + (1-\alpha)t_2)}{\alpha t_1 + (1-\alpha)t_2} .$$

Since $M(t)$ is convex,

$$H(\alpha t_1 + (1-\alpha)t_2) \leq \frac{\alpha M(t_1) + (1-\alpha)M(t_2)}{\alpha(t_1) + (1-\alpha)t_2} .$$

Observe that if $x > 0,\ y > 0$ then

$$\left(\frac{b}{y} \leq \frac{a}{x}\right) \Leftrightarrow bx \leq ay \Leftrightarrow (a+b)x \leq a(x+y) \Leftrightarrow \frac{a+b}{x+y} \leq \frac{a}{x} .$$

Suppose $H(t_1) = \frac{M(t_1)}{t_1} \geq \frac{M(t_2)}{t_2}$.

With the substitutions $a = \alpha M(t_1)$, $x = \alpha t_1$, etc. The conclusion is:

$$H(\alpha t_1 + (1-\alpha)t_2) \leq H(t_1) = \max\ (H(t_1),\ H(t_2)),$$

i.e. the function $H(t)$ is quasi-convex.

Q.E.D.

Remark

Since the function $M(t)$ is defined and convex on $[0, +\infty)$ it is continuous for all $t \in (0, +\infty)$. The function $H(t) = \frac{M(t)}{t}$ is defined for all $t > 0$ and it is the ratio of two continuous functions, hence, $H(t)$ is continuous function for all $t > 0$.

Proof of Theorem 4

For definition (28)

$$H(t) = \underset{y \in T(t)}{\text{minimum}} \frac{N(y,t)}{t}$$

$$= \text{minimum} \frac{1}{t} \sum_{j=1}^{n} \left[K_j + \frac{h_j}{2R_j} y_j^{\,2} + \frac{\bar{P}_j}{2R_j} (R_j t - y) + (P_j + c_j - r_j) R_j t - P_j y_j \right]$$

since $y_j \leq R_j t$ $(j=1, \ldots, n)$

$$\lim_{t \to 0+} H(t) = \begin{cases} \text{bounded if } \sum_{j=1}^{n} K_j = 0 \\ +\infty \quad \text{if } \sum_{j=1}^{n} K_j > 0 . \end{cases}$$

In the first case define $H(0) = \lim_{t \to 0+} H(t)$.

Then the function $H(t)$ is continuous for all $t \geq 0$. In Theorem 3, it was shown that the optimal review period t^* of program B' is bounded by t^+ . Hence, since $H(t)$ is continuous on the compact set $[0, t^+]$, the theorem follows immediately.

If $H(t) \underset{t \to 0+}{\to} +\infty$, one can find a number $\xi > 0$ such that $H(t) > H(t^+)$ for $t < \xi$. But $H(t)$ is continuous on the compact set $[\xi, t^+]$. Hence $H(t)$ has an absolute minimum on $[\xi, t^+]$.

Q.E.D.

In order to find the optimal solution of program B' an algorithm is given. The algorithm is based mainly on the following:

(i) For each $t > 0$, the program

$$\text{Minimize} \quad N'(y,t)$$
$$\text{Subject to} \quad y \in T(t)$$

always has an optimal solution.

(ii) $H(t)$ is a continuous (strictly) quasi-convex function of t.

3. Algorithm

Step 0

(a) Find the upper bound t^+

(b) Let $So = \{0, t^+\}$

Step 1

Find t_i such that $H(t_i) = \min_{t \in S_i} H(t)$

Step 2

(a) Find t' and t'' such that

$$t' = \min_{t \in S_i} [(t_i - t) > 0]$$

$$t'' = \begin{cases} \min_{t \in S_i} [(t - t_i) > 0] & \text{if} \quad t_i < t^+ \\ t^+ & \text{if} \quad t_i = t^+ \end{cases}$$

(b) Let $t'_i = \frac{1}{2}(t' + t_i)$ and $t''_i = \frac{1}{2}(t'' + t_i)$

Step 3

Calculate $H(t'_i)$ and $H(t''_i)$

Setp 4

Let $S_{i+1} = S_i \cup \{t'_i , t''_i\}$

and return to Step 1, with $i+1$ replacing i .

In Step 0, the upper bound t^+ is found by solving the program (minimize $N(y,t)$ subject to $(y,t) \varepsilon T$), see Theorem 3.

In Step 3, the values of $H(t'_i)$ and $H(t''_i)$ are found in accordance with the formulas (28) and (29).

It must be noted that at each iteration it is necessary to solve the following programs

$$\begin{array}{llll} \text{Minimize} & N'(y,t'_i) & \text{and Minimize} & N'(y,t''_i) \\ \text{Subject to} & y \ \varepsilon \ T(t'_i) & \text{Subject to} & y \ \varepsilon \ T(t''_i) . \end{array}$$

Define

$$H(0) = \lim_{t \to 0+} H(t), \text{ and}$$

$$H(t^*) = \min_{t \varepsilon [0,t^+]} H(t)$$

Theorem 7

At the K^{th} $(K=1, 2, \ldots)$ iteration it is true that

(a) $t_K - t' = \dfrac{t^+}{2^K}$ and $t'' - t_K = \begin{cases} \dfrac{t^+}{2^K} & \text{if } t_K < t^+ \\ 0 & \text{if } t_K = t^+ \end{cases}$

(b) $|t^* - t_K| \leq \dfrac{t^+}{2^K}$.

Proof:

By induction over K,

(i) the theorem is clearly true for $K=1$

(ii) assume that it is true for $K=r$ and consider first, the case $t_r < t^+$.

Then, at the r^{th} iteration

(32) $$t_r - t' = \frac{t^+}{2^r}\ ,\ t'' - t_r = \frac{t^+}{2^r} \text{ and } |t^o - t_r| \leq \frac{t^+}{2^r}\ ,$$

where

(33) $$t'_r = \frac{1}{2}(t' + t_r) \text{ and } t''_r = \frac{1}{2}(t'' + t_r)\ .$$

$$S_{r+1} = S_r \cup \{t'_r, t''_r\}\ .$$

At the $(r+1)^{th}$ iteration

Step 1

$$H(t_{r+1}) = \min H(t) = \min(H(t_r), H(t'_r), H(t''_r))$$

(34) Suppose that $t_{r+1} = t_r$ (i.e. $H(t_r) < H(t'_r), H(t''_r)$)

Step 2

(35) $$t' = t'_r,\ t'' = t''_r\ .$$

Then, from (32), (33), (34), and (35),

$$t_{r+1} - t' = t_r - t'_r = t_r - \frac{1}{2}(t' + t_r) = \frac{1}{2}(t_r - t') = \frac{1}{2} \cdot \frac{t^+}{2^r} = \frac{t^+}{2^{r+1}}\ ,$$

$$t'' - t_{r+1} = t''_r - t_r = \frac{1}{2}(t'' + t_r) - t_r = \frac{1}{2}(t'' - t_r) = \frac{1}{2} \cdot \frac{t^+}{2^r} = \frac{t^+}{2^{r+1}}\ .$$

Hence, the part (a) of the theorem is true for all $K=1, 2, \ldots$

Also, $t^* \varepsilon [t'_r, t''_r]$, because if $t^* < t'_r$, then there exist $\lambda > 0$, such that $(\lambda t^* + (1-\lambda)t_r) = t'_r$ and $H(\lambda t^* + (1-\lambda)t_r) = H(t'_r)$.

But the quasi-convexity of $H(t)$ implies that $H(t'_r) =$ $= H(\lambda t^* + (1-\lambda)t_r) < \max(H(t^*), H(t_r)) = H(t_r)$, which contradicts the assumption (34). Hence, $t'_r \leqq t^*$.

In a similar way the assumtion $t''_r < t^*$ also leads to a contradiction. Hence $t^* \varepsilon [t'_r, t''_r]$. This result and the formulas (32) and (33) imply that

$$|t^*-t_{r+1}| = |t^*-t_r| \leqq |t'_r-t_r| = |\tfrac{1}{2}(t'+t_r)-t_r| =$$
$$= \tfrac{1}{2}|t_r-t'| = \tfrac{1}{2} \cdot \frac{t^+}{2^r} = \frac{t^+}{2^{r+1}} .$$

Hence, $|t^*-t_{r+1}| \leqq \dfrac{t^+}{2^{r+1}}$. Then part (b) of the theorem is true for all positive integer $K=1, 2, \ldots$

A similar analysis for the cases $t_{r+1} = t'_r$ or t''_r in (34), completes the proof with the assumption that $t_r < t^+$.

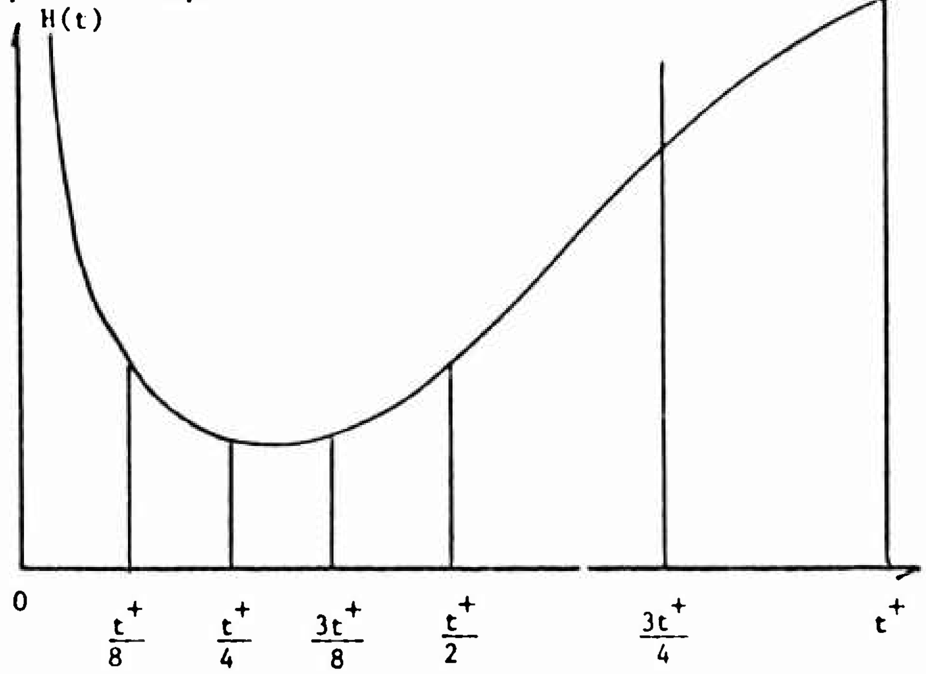
Following the same steps, the analysis of the case $t_r = t^+$, completes the proof.

Corollary 1

$$\lim_{K\to\infty} H(t_K) = H(t^*)$$

The corollary follows immediately from the facts: $H(t)$ is a continuous function and $\lim_{K\to\infty} t_K = t^*$.

Graphical example



Iteration 0

(a) Find the upper bound t^+

(b) $S_o = \{0, t^+\}$

Step 1

$$H(t^+) = \min_{t \in S_o} H(t), \quad t_o = t^+$$

Step 2

(a) $t' = 0, \; t'' = t^+$

(b) $t'_o = \frac{t^+}{2}, \; t''_o = t^+$

Step 3

Calculate $H\left(\frac{t^+}{2}\right)$ and $H(t^+)$

Step 4

$$S_1 = S_o \cup \left\{\frac{t^+}{2}, t^+\right\} = \left\{0, \frac{t^+}{2}, t^+\right\}, \text{ return to Step 1.}$$

Iteration 2

Step 1

$$H(t_1) = \min_{t \in S_1} H(t) = H\left(\frac{t^+}{2}\right), \; t_1 = \frac{t^+}{2}$$

Step 2

(a) $t' = 0, \; t'' = t^+$

(b) $t'_1 = \frac{1}{2}\left(0 + \frac{t^+}{2}\right) = \frac{t^+}{4} \; ; \; t''_1 = \frac{1}{2}\left(t^+ + \frac{t^+}{2}\right) = \frac{3}{4} t^+ .$

Step 3

Calculate $H(t'_1)$ and $H(t''_1)$.

Step 4

$$S_2 = S_1 \cup \{t'_1, t''_1\} = \left\{0, \frac{t^+}{4}, \frac{t^+}{2}, \frac{3t^+}{4}, t^+\right\}, \text{ return to Step 1.}$$

Iteration #2

Step 1

$$H(t_2) = \min_{t \in S_2} H(t) = H\left(\frac{t^+}{4}\right), \; t_2 = \frac{t^+}{4}$$

Step 2

(a) $t' = 0, \; t'' = t_1 = \frac{t^+}{2}$

(b) $t'_2 = \frac{1}{2}(0+t_2) = \frac{t^+}{8}, \; t''_2 = \frac{1}{2}\left(\frac{t^+}{2} + \frac{t^+}{4}\right) = \frac{3t^+}{3} .$

Step 3

Calculate $H(t'_2)$ and $H(t''_2)$

Step 4

$$S_3 = S_2 \cup \{t'_2, t''_2\} = \left\{0, \frac{t^+}{8}, \frac{t^+}{4}, \frac{3t^+}{8}, \frac{t^+}{2}, \frac{3t^+}{4}, t^+\right\}, \text{ return to Step 1.}$$

etc.

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